九十七學年度國立交通大學應用數學系博士班入學考

考試科目: 分析

1. (16 points) Let $\mathcal{B}^n$ be the collection of all Borel sets in $\mathbb{R}^n$ and let $\mathcal{L}^n$ be the collection of all Lebesgue measurable sets in $\mathbb{R}^n$. Define the function $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ by

$$f(x) = (x, 0) \qquad \text{for all } x \in \mathbb{R}.$$

   (a) (8 points) Is f $\mathcal{B}^1$-$\mathcal{B}^2$-measurable? Justify your answer.

   (b) (8 points) Is f $\mathcal{L}^1$-$\mathcal{L}^2$-measurable? Justify your answer.

2. (24 points) Consider a sequence of functions $(f_n)_{n\in\mathbb{N}}$ defined on $[0,1]$ by setting

$$f_n(x) = \frac{nx}{1+n^2x^2} \qquad \text{for } x \in [0,1].$$

   (a) (12 points) Show that $(f_n)_{n\in\mathbb{N}}$ is a uniformly bounded sequence on $[0,1]$ and evaluate

$$\lim_{n\to\infty} \int_{[0,1]} \frac{nx}{1+n^2x^2}\, dx.$$

   (b) (12 points) Justify if $(f_n)_{n\in\mathbb{N}}$ is uniformly convergent on $[0,1]$.

3. (12 points) Define $f : [0,1] \longrightarrow [0,1]$ by

$$f(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \cap [0,1] \\ 2^{n_x}, & \text{if } x \in \mathbb{Q}^c \cap [0,1], \end{cases}$$

where n_x is the number of leading zeros in the decimal expansion of x, i.e., for $x = \sum_{k=1}^{\infty} \frac{a_k}{10^k}$ (with $a_k = 0, 1, 2, ..., 9$),

$$n_x = \begin{cases} 0, & \text{if } a_1 \neq 0, \\ \inf\{k \geq 0 : a_1 = \cdots = a_k = 0, a_{k+1} \neq 0\}, & \text{othersie.} \end{cases}$$

Show that f is measurable, and find $\int_0^1 f(x)\, dx$.

4. (24 points) Let m be the Lebesgue measure on $\mathbb{R}^n$ and let f be a measurable function on $X \subset \mathbb{R}^n$. Suppose that $\int_X |f(x)|^p \, dx < \infty$ for some $p \in (0, \infty)$.

(a) (12 points) Show that

$$\lim_{\lambda \to \infty} \lambda^p m(x \in X : |f(x)| > \lambda) = 0.$$

(b) (12 points) Show that

$$\int_X |f(x)|^p \, dx = \int_{[0,\infty)} p\lambda^{p-1} m(x \in X : |f(x)| > \lambda) \, d\lambda.$$

5. (24 points) Let f and g be two functions defined on $[0, 1]$ by

$$f(x) = \sqrt{x} \qquad \text{for } x \in [0, 1],$$

and

$$g(x) = \begin{cases} x^2 \left| \sin\left(\frac{1}{x}\right) \right|, & \text{for } x \in (0, 1], \\ 0, & \text{for } x = 0. \end{cases}$$

(a) (12 points) Is $g \circ f$ absolutely continuous on $[0, 1]$? Justify your answer.

(b) (12 points) Is $f \circ g$ absolutely continuous on $[0, 1]$? Justify your answer.