# Ph.D. Entrance Exam: Linear Algebra
May 22, 2007

1. (10 points) Determinate whether

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

   are the matrices of the same linear transformation in two different pairs of bases. Justify your answer.

2. (10 points) Prove that the matrix

$$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 0 & 2 & 1 \\ 1 & 2 & 0 & 2 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

   has two positive and two negative characteristic roots (eigenvalues), taking into account multiplicities.

3. Let $[A, B] = AB - BA$ be the commutator product of $n \times n$ real matrices $A$ and $B$.

   (a) (5 points) Show that for any $2 \times 2$ real matrices $A$, $B$ and $C$, $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = O$, where $O$ is the zero matrix.

   (b) (5 points) Show that for any $2 \times 2$ real matrices $A$ and $B$, $[A, B]^2 = aI$, for some $a \in \mathbf{R}$, where $I$ is the identity matrix.

   (c) (5 points) Let $C$ be a $2 \times 2$ real matrix. Show that $tr(C) = 0$ if and only if $C = [A, B]$ for some $2 \times 2$ real matrices $A$ and $B$.

   (d) (5 points) Are the statements $(a)$ and $(b)$ valid for $n \times n$ real matrices ? Either prove or give a counterexample.

4. *Prove or give a counterexample*:

   (a) (5 points) If $A$ is an $n \times n$ real matrix such that $A^2 = A$, then $tr(A)$ is an integer.

   (b) (5 points) If $A$ and $B$ are $n \times n$ real matrices, then the minimal polynomials for $AB$ and $BA$ are equal.

   (c) (5 points) If $A$ and $B$ are $n \times n$ real matrices, $B$ is not the zero matrix, then $det(A + xB) = 0$, for some real number $x$.

5. Let $\mathbf{V}$ be an n-dimensional vector space over $\mathbf{R}$, and let $T$ be a linear transformation of $\mathbf{V}$ into itself such that the range $T(\mathbf{V})$ and null space (kernel) $N(T)$ are identical, $T(\mathbf{V}) = \mathbf{N}(\mathbf{T})$ .

   (a) (5 points) Show that $n$ is even.

   (b) (5 points) What can you say about the characteristic polynominal and the minimal polynominal of $T$.

   (c) (5 points) Give an example of such a linear transformation.

6. Let $\mathbf{V}$ be an n-dimensional vector space over $\mathbf{R}$. Suppose that $J$ is a linear transformation of $\mathbf{V}$ into itself satisfying $J^2 = -I$, where $I$ is the identity mapping.

   (a) (5 points) For $a+ib \in \mathbf{C}$ and $u \in \mathbf{V}$, define $(a+ib) \cdot u = au + bJ(u)$, show that $\mathbf{V}$ is a vector space over $\mathbf{C}$.

   (b) (5 points) What is the dimension of $\mathbf{V}$ if we consider $\mathbf{V}$ as a vector space over $\mathbf{C}$ of $(a)$ ? Justify your answer.

   (c) (5 points) If $v_1, v_2, \cdots, v_m$ is a basis for $\mathbf{V}$ over $\mathbf{C}$ of $(a)$, can you construct a basis for $\mathbf{V}$ over $\mathbf{R}$ ?

7. (15 points) Let $\mathbf{V}$ be a finite dimensional vector space over $\mathbf{R}$, and $T$ is a linear transformation of $\mathbf{V}$ into itself. Suppose that the characteristic polynominal $p_T(x)$ of $T$ is written as $p_T(x) = p_1(x)p_2(x)$, where $p_1(x)$ and $p_2(x)$ are two relatively prime polynominals with real coefficients. Show that every vector $v \in \mathbf{V}$ can be written in a unique way as $v = v_1 + v_2$, where $v_1, v_2 \in \mathbf{V}$, $p_1(T)(v_1) = 0$ and $p_2(T)(v_2) = 0$.