96 學年度 應用數學系 博士班招生 「離散數學」考題

1. (20 points) In a city with a regular *chessboard* street palne, the North-South streets are called 1st Street, 2nd Street, …, 20th Street, and the East-West streets are called 1st Avenue, 2nd Avenue, …, 10th Avenue. What is the minimum number of blocks you have to walk to get from the corner of 1st Street and 1st Avenue to the corner of 20th Street and 10th Avenue? In how many ways can you get there walking this minimum number of blocks?

2. (20 points) Let X, Y be finite sets, discuss and find the numbers of
   a. all functions from X into Y;
   b. all one to one functions from X into Y; and
   c. all onto functions from X into Y respectively.

3. (20 points) Let M be a (0,1)-matrix with a constant column sum k, and a constant row sum r,
   a. give an explanation of the fact that $MM^t = rI + \lambda(J - I)$;
   b. find the determinant of $MM^t = rI + \lambda(J - I)$, and then derive possible information?

4. (20 points)
   Let A, M be the adjacency matrix and the incidence matrix of a graph G respectively,
   a. show that $M^tM = A_L + 2I$ where A_L is the adjacency matrix of the line graph $L(G)$ of G;
   and $MM^t = A + kI$ if G is k-regular;
   b. find $\det(UV)$ where $U = \begin{bmatrix} xI_n & -M \\ 0 & I_m \end{bmatrix}$ and $V = \begin{bmatrix} I_n & M \\ M^t & xI_m \end{bmatrix}$, derive possible information?

5. (20 points) A professor decides to play a little game:
   "*I bet that there are two of you who have the same birthday! What do you think*?"
   Several students reply immediately:
   "*There are 366 possible days a year, so you could only conclude that if there were at least 367 of us in the class! But there are only 50 of us, and so you would lose the bet.*"
   Nevertheless, the professor insists on betting:

   a. explain that the professor will definitely win the bet if the class has at least 367 students.
   b. find a rational function $f(n,k)$ so that $f(366,50)$ is the probability that the professor win the bet if the class has 50 students ?

   c. estimate $f(n,k)$ by finding its bounds?

   (the fact $\frac{x-1}{x} \le \ln x \le x-1$ for $x > 0$ may be useful.)

   d. find k so that $f(366,k) \ge 1/2$?