Ph.D. Entrance Exam: Analysis
May 16, 2006

1. (20%) Consider the system of equations $\begin{cases} \ln(xu) + e^{yv^2} = 3x, \\ \sin(xv) + \dfrac{yu}{y^2 + u^2} = 5y. \end{cases}$ Is
   the system uniquely solvable for u, v in terms of x, y near $(x, y, u, v) = (1, -\pi, \pi, -1)$? Why?

2. Let $f : (0, 1) \to \mathbb{R}$ be a differentiable function.

   (a) (10%) Prove or disprove that the derivative function f' has the intermediate value property on $(0, 1)$.

   (b) (10%) Prove or disprove that f' is continuous on $(0, 1)$.

3. Consider a real-valued function f on an interval.

   (a) (10%) Show that if f satisfies a Lipschitz condition then f is absolutely continuous.

   (b) (10%) Suppose that f is absolutely continuous and differentiable so that f' is integrable. Show that f satisfies a Lipschitz condition if and only if $|f'|$ is bounded.

4. (20%) Let $f_n : [0, 2] \to \mathbb{R}$ be a sequence of differentiable functions whose derivatives are uniformly bounded. Prove that if $f_n(1)$ is bounded as $n \to \infty$, then the sequence (f_n) has a subsequence that converges uniformly on $[0, 2]$.

5. (20%) Let X be a linear vector space that is complete in the norms $\| \cdot \|_1$ and $\| \cdot \|_2$. Prove that if there is a constant $C > 0$ such that $\|x\|_1 \leq C\|x\|_2$ for all $x \in X$, then the norms are equivalent.