94学年度博士班入学考試 —線性代數

1. Let

$$A=\begin{bmatrix} \frac{4+\sqrt{3}}{4} & \frac{3}{4} \\ -\frac{1}{4} & \frac{4-\sqrt{3}}{4} \end{bmatrix}.$$

   (a) (6 points) Find the characteristic polynomial of A.

   (b) (14 points) Compute A^{2005} and A^{-5}.

2. (20 points) Let $M_{2\times 2}$ be the vector space of all real 2×2 matrices, let

$$A=\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix},$$

$$B=\begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix},$$

and define a linear transformation $L: M_{2\times 2} \longrightarrow M_{2\times 2}$ by $L(X)=AXB$.

   (a) (10 points) Compute the trace and determinate of L.

   (b) (10 points) Find the rank of L.

3. (a) (10 points) Let

$$A=\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

   Find $\frac{1}{0!}A^0+\frac{1}{1!}A^1+\frac{1}{2!}A^2+\cdots\frac{1}{n!}A^n+\cdots$.

   (b) (10 points) Let A be a real $n\times n$ matrix. Is the series $\sum_{k=0}^{\infty}\frac{1}{k!}A^k$ convergent entry-wise ?

4. Let A and B be real $n\times n$ matrices.

   (a) (6 points) Let v be a characteristic vector (eigenvector) of AB. Is v a characteristic vector (eigenvector) of BA ?

   (b) (7 points) Are there such matrices A and B with $AB-BA=I$? where I is the identity matrix.

   (c) (7 points) Suppose that both A and B are diagonalizable, when does there exist an invertible real $n\times n$ matrix C such that both CAC^{-1} and CBC^{-1} are diagonal matrices ?

5. For all the following statements, if the statement is true, explain why; if the statement is false, give a counter example.

   (a) (6 points) If A and B are $n\times n$ nilpotent matrices, then $A+B$ is a nilpotent matrix. (An $n\times n$ matrix C is nilpotent if $C^k=O$ for some positive integer k.)

   (b) (7 points) If A and B are real $n\times n$ matrices, A and B have the same characteristic and minimal polynomial, then A and B are similar.

   (c) (7 points) If A and B are real $n\times n$ matrices, rank$A$ = rank$B$, $A^2=2A$, $B^2=2B$, then there is an invertible real matrix D such that $DAD^{-1}=B$.