博士班入學考試
離散 P1

Discrete Mathematics
Entrance Examination for Ph.D. Program
Department of Applied Mathematics, National Chiao Tung University
May 31, 2002

1. (a) (4%) Given 20 Chinese books, 8 English books, 15 Japanese books, 25 French books, and 10 German books, how many books must be chosen to *guarantee* there are 13 books of *the same language*?

(b) (4%) Twenty-one persons have first names Alex, Jenny, Sonia, and Sonia and last names Dumont, Fiol, Hsu, Chazelle, and Chung. Prove that *at least two* persons have *the same first and last names.*

(c) (7%) Suppose the numbers $1, 2, \cdots, 20$ are randomly positioned around a *circle*. Prove that no matter how these 20 numbers are positioned around the circle, there *always exist three consecutive numbers* such that *the sum* of these three consecutive numbers is *at least 32.*

2. Let $a_1, a_2, \cdots, a_{10}$ denote the ages of 10 persons. Let $r_i = a_i \pmod{16}$ and let

$$s_i = \begin{cases} r_i & \text{if } r_i \leq 8 \\ 16 - r_i & \text{it } r_i > 8. \end{cases}$$

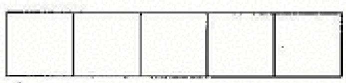
(a) (2%) Prove that each value of $s_1, s_2, \cdots, s_{10}$ is contained in $\{0, 1, \cdots, 8\}$.

(b) (3%) Prove that if $r_j > 8$ and $r_k > 8$, then $16|a_j - a_k$ (i.e., 16 divides $a_j - a_k$).

(c) (3%) Prove that if $s_j = r_j$ and $s_k = 16 - r_k$, then $16|a_j + a_k$ (i.e., 16 divides $a_j + a_k$).

3. (a) (3%) Solve the recurrence relation $a_n - 4a_{n-1} + 4a_{n-2} = 0$, $(n \geq 2)$ with initial conditions $a_0 = 1$ and $a_1 = 3$.

(b) (7%) Let b_n denote the number of different ways to color the squares of a 1-by-n board with the colors *red*, *white*, and *blue* in which *no adjacent squares are colored red simultaneously.* Find a recurrence relation for b_n and justify it. The following is an example of a 1-by-5 board with 5 squares.



(c) (3%) Let c_n be the number of nonnegative integral solutions of

$$2x_1 + 5x_2 + x_3 + 7x_4 = n.$$

Determine the *generating function* for c_n (but need not to determine c_n).

(d) (7%) Let d_n be the number of ways to color the squares of a 1-by-n board with the colors red, white, and blue where the number of red squares is *even* and the number of blue squares is *at least one. Use exponential generating function* to determine the number d_n.

4. Ackermann's function is defined by the recurrence relations

$$\begin{aligned} A(m,0) &= A(m-1,1), & m &= 1,2,\cdots, \\ A(m,n) &= A(m-1,A(m,n-1)), & m &= 1,2,\cdots, \\ & & n &= 1,2,\cdots, \end{aligned}$$

and the initial conditions $A(0,n) = n+1, \quad n = 0,1,\cdots$.

(a) (2%) Compute the value of $A(1,1)$.

(b) (4%) *Use induction* to prove that $A(1,n) = n+2$, for all $n \geq 0$.

(c) (4%) *Use induction* to prove that $A(2,n) = 3+2n$, for all $n \geq 0$.

(d) (7%) *Use induction* to prove that $A(3,n) > n$, for all $n \geq 0$.

5. Let $S_{n,k}$ denote the number of ways to seat n children at k round tables, each table with at least one child. Note that the ordering of the tables is not taken into account. Also note that the seating arrangement at a table is regarded as a circular permutation.

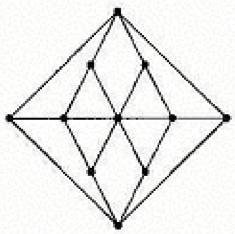
(a) (2%) Prove that $S_{n,1} = (n-1)!$ for all $n \geq 1$.

(b) (3%) Prove that $S_{n,n-1} = \binom{n}{2}$ for all $n \geq 1$.

(c) (7%) Prove that $\sum_{k=1}^{n} S_{n,k} = n!$ for all $n \geq 1$.

6. **(a)** (7%) Prove that a simple graph with n vertices and with $\binom{n-1}{2} + 1$ edges is *connected*.

(b) (7%) Does the following graph H has a *Hamiltonian cycle*? Justify your answer.



(c) (7%) Find the *chromatic number* of the graph H in **(b)**. Justify your answer.

(d) (7%) Let $G = (V, E)$ be a *color-critical* graph, i.e., each subgraph of G obtained by removing a vertex has a smaller chromatic number. Prove that

$$\chi(G_{V-\{x\}}) = \chi(G) - 1 \quad \text{for all } x \in V,$$

where $G_{V-\{x\}}$ is the subgraph of G obtained by removing the vertex x.