Department of Applied Mathematics, National Chiao Tung University

1998-6-3

1. A *composition* of a positive integer n is an expression of n as an *ordered* sum of positive integers. For instance, there are eight compositions of 4; namely,

$$4 \qquad 3+1 \qquad 2+2 \qquad 1+3 \qquad 2+1+1 \qquad 1+2+1 \qquad 1+1+2 \qquad 1+1+1+1.$$

The *Fibonacci numbers* are defined by $F_1 = 1, F_2 = 1, F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. Express the following numbers in terms of the Fibonacci numbers. Justify your answers.

(a) The number a_n of subsets S of the set $\{1, 2, \ldots, n\}$ such that S contains no two consecutive integers.

(b) The number b_n of compositions of n into parts greater than 1.

(c) The number c_n of sequences $(r_1, r_2, \ldots, r_n)$ of 0's and 1's such that $r_i \geq r_{i-1}$ and $r_i \geq r_{i+1}$ for all even i.

(d) $d_n = \sum s_1 s_2 \ldots s_k$, where the sum is over all compositions $s_1 + s_2 + \ldots + s_k = n$.

(e) The number e_n of sequences $(t_1, t_2, \ldots, t_n)$ of 0's, 1's and 2's such that 0 is never followed immediately by 1.

2. Suppose G is a graph with n vertices.

(a) Prove that if G is disconnected, then its complement G^c is connected.

(b) Find at least three examples of graphs G such that both G and G^c are connected.

(c) Prove that if G contains no P_4 as an induced subgraph, then either G or G^c is disconnected.

(d) Prove that if the minimum degree $\delta(G) \geq (n-1)/2$, then G is connected.

3. For a positive integer n, consider the graph $G_n = (V_n, E_n)$ with vertex set V_n the set of all $n \times n$ binary matrices, and two vertex A and B are adjacent if and only if they are the same except a certain row of A is the complement of the corresponding row of B. For instance, in G_3,

$$\text{vertex } A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ is adjacent to vertex } B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

as the first and the last rows of these two matrices are the same and the second rows are complement to each other.

(a) Determine the following parameters for the graph G_n: the number of vertices, the number of edges, the number of (connected) components and the diameters of all components.

(b) Define the graph $H_n = (V_n, F_n)$ in a similar way as G_n except that two vertices A and B are adjacent if and only if they are the same except a certain row (or column) of A is the complement of the corresponding row (or column) of B. For instance, in H_3,

$$\text{vertex } A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ is adjacent to vertices } B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Determine the following parameters for the graph H_n: the number of vertices, the number of edges, the number of (connected) components and the diameters of all components.

4. Let $x_1, x_2, \ldots, x_n$ be a sequence of real numbers (not necessarily positive). Consider the problem of finding a subsequence $x_i, x_{i+1}, \ldots, x_j$ (of consecutive elements) such that the product of the numbers in it is maximum over all consecutive subsequences. The product of the empty subsequence is defined as 1. One only needs to output the maximum product rather than the corresponding subsequence.

(a) Consider the following algorithm that solves the problem.

```
answer <- 1;
for i = 1 to n do
for j = i to n do
      p <- 1;
      for k = i to j do p <- p * x_k end do;
      if p > answer then answer <- p;
      end do;
      end do;
output answer.
```

What is the time complexity of the above algorithm.

(b) Write an algorithm for the problem with a better time complexity than that in part (a). Most favorable one is of $O(n)$ time.

(c) Do the same problem except the subsequence is only required to be the form $x_{i_1}, x_{i_2}, \ldots, x_{i_j}$ with $1 \le i_1 < i_2 < \ldots < i_j \le n$.

(d) Do the same problem except that the product is changed to the sum. (Of course, negative terms are still allowed.) The sum of the empty subsequence is defined as 0.

5. Suppose S is a finite nonempty set and 2^S the power set of S, namely, $2^S = \{A : A \subseteq S\}$. A *matroid* on S is an ordered pair $(S, \mathcal{I})$ with $\mathcal{I} \subseteq 2^S$ such that the following three conditions hold.

(I1) $\emptyset \in \mathcal{I}$.

(I2) If $A \subseteq B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$.

(I3) If $A, B \in \mathcal{I}$ and $|A| = |B| + 1$, then there exists an element $x \in A \setminus B$ such that $B \cup \{x\} \in \mathcal{I}$.

In this definition, an element of $\mathcal{I}$ is called an *independent set* of the matroid.

(a) For integers $n \ge k \ge 0$, let $S_n = \{1, 2, \ldots, n\}$ and $\mathcal{I}_{n,k} = \{A \subseteq S_n : |A| \le k\}$. Determine all n and k for which $(S_n, \mathcal{I}_{n,k})$ is a matroid. Justify your answer.

(b) The *basis* of a matroid is an independent set that is not a proper subset of any other independent set. Prove that any two bases of a matroid have the same cardinality.

(c) Find all bases of those $(S_n, \mathcal{I}_{n,k})$, as in part (b), that are matroids.

(d) The *rank function* of a matroid $(S, \mathcal{I})$is a function ρ from 2^S to the set of all nonnegative integers defined by $\rho(A) = \max\{|I| : I \subseteq A \text{ and } I \in \mathcal{I}\}$. Prove that for any matroid the following hold.

(R1) $\rho(\emptyset) = 0$.

(R2) $\rho(A) \le \rho(A \cup \{x\}) \le \rho(A) + 1$ for any $A \subseteq S$ and $x \in S$.

(R3) If $\rho(A \cup \{x\}) = \rho(A \cup \{y\}) = \rho(A)$, then $\rho(A \cup \{x, y\})$.

(e) Determine the ranks of all sets for those $(S_n, \mathcal{I}_{n,k})$, as in part (b), that are matroids.