Numerical Analysis

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1. Give a function $g: R^n \to R^n$.

   (a) (5%) Under what conditions the function g is called a contraction mapping?

   (b) (5%) State the contraction-mapping theorem.

   (c) (10%) Given equations $x^2 + y^2 - 1 = 0$ and $5x^2 + 21y^2 - 9 = 0$, prove that There exists a solution in the first quadrant of the (x,y) coordinate system by using the contraction-mapping theorem.

2. Suppose a function f has a continuous fourth derivative on the interval [-1,1].

   (a) (8%) Construct the Hermite interpolation polynomial of degree 3 by using the interpolation points at $x=-1$ and $x=1$.

   (b) (12%) Prove that

$$\left|\int_{-1}^{1} f(x)\,dx - \left[f(-1)+f(1)\right] - \frac{1}{3}\left[f'(-1)-f'(1)\right]\right| \le \frac{2}{45}\max_{x\in[-1,1]}\left|f^{(4)}(x)\right|$$

3. Suppose you are solving an initial value problem $u'(t) = f(t,u(t))$ with an initial value $u(0) = u_0$ by a multi-step method $u_{n+1} = \sum_{j=0}^{p} a_j u_{n-j} + h\sum_{j=0}^{p} b_j f_{n-j} + hb_{-1} f_{n+1}$ where h is the time step. It is well known that a consistent multi-step method converges if and only if it is zero-stable and the error on the initial u_0 tends to zero as $h \to 0$.

   (a) (6%) Describe the definition of consistency and zero-stability.

   (b) (4%) Give a necessary and sufficient condition for determining a given scheme is consistent.

   (c) (4%) Give a necessary and sufficient condition for determining a given scheme is zero-stable.

(d) (6%) Given a linear multi-step method, $u_{n+1} = \alpha u_{n-1} + \frac{h}{3}\left[f_{n+1} + 4f_n + f_{n-1}\right]$,

determine the set of all value α such that the scheme is zero-stable. Find the value α such that the order of accuracy of the scheme is as high as possible.

4.

(a) (10%) Find the QR factorization of the matrix $H = \begin{bmatrix} 9 & -6 \\ 12 & -8 \\ 0 & 20 \end{bmatrix}$ and find the

least square solution of the equation $Hx = b$, where $b = \begin{bmatrix} 300 & 600 & 900 \end{bmatrix}^T$.

(b) (10%) The matrix H in part (a) is an upper-Hessenberg matrix. In general a matrix H is called upper-Hessenberg if $H = \left(h_{ij}\right)_{m\times n}$, where $h_{ij} = 0$ for $i > j+1$.

Given a matrix A, describe the detail steps to turn the matrix A into an upper Hessenberg matrix via a sequence of similarity transforms?

5. (20%) Given the heat equation $u_t = cu_{xx}$ where $c > 0$ and $x \in [0,1]$, with initial data $u(x,0) = g(x)$ and boundary data $u(0,t) = u(1,t) = 0$, suppose we numerically solve the equation using Crank-Nicolson method.

(a) (4%) Write down the Crank-Nicolson scheme.

(b) (16%) Prove that the scheme is unconditionally stable and is second order accurate in both space and time.