102 學年度 國立交通大學 應用數學系

博士班入學考 線性代數 試題

2013/05/07

(10%) 1. Let A and B be linear transformations on a finite-dimensional vector space   Prove that dim ker $AB \leq$ dim ker A + dim ker B.

(10%) 2. Let V be the inner-product space consisting of all real polynomials p of degree at most 2 on [0, 1] under the inner product $\langle p, q \rangle = \int_0^1 p(x)q(x)dx$. Find an orthonormal basis of V.

(15%) 3. (a) Prove that every n-by-n complex matrix A is similar to its transpose A^T. (10%)

(b) Is every complex A similar to its adjoint A^*? Either prove it or give a counterexample. (5%)

(15%) 4. Let A be an n-by-n real symmetric, tridiagonal matrix of the form

$$\begin{bmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & \ddots & \ddots & \\ & & \ddots & \ddots & b_{n-1} \\ & & & b_{n-1} & a_n \end{bmatrix}$$ with all b_j's nonzero.

(a) Prove that A has n distinct eigenvalues. (10%)

(b) From (a), what can you say about dim ker A and rank A? (5%)

(20%) 5. Let $A = \begin{bmatrix} 3 & 1 \\ -1 & 1 \end{bmatrix}$.

(a) Find the Jordan canonical form B of A and find a 2-by-2 invertible matrix X such that $X^{-1}AX = B$. (10%)

(b) Use the result in (a) to calculate A^{100} and A^{-8} (10%)

(30%) 6. Let A be an n-by-n complex Hermitian matrix

(a) Prove that all eigenvalues of A are real. (10%)

(b) Prove that the maximum eigenvalue of A equals max $\{Ax \cdot x : x \in \mathbb{C}^n, \|x\| = 1\}$. (10%)

(c) Let A be written as $\begin{bmatrix} B & C \\ C^* & D \end{bmatrix}$. Use (b) to prove that the maximum eigenvalue of B is less than or equal to the maximum eigenvalue of A. (10%)