USRP 2025: REPRESENTATIONS DISTINGUISHED BY SUBGROUPS

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1. Motivation and introduction

Branching problems in representation theory is the broad problem of understanding how irreducible representations of a group behave when restricted to a subgroup. To be more precise, let (π, V) be an irreducible representation of a group G. Let H be a subgroup of G and (σ, W) be an irreducible representation. A basic question is to understand the space $\mathrm{Hom}_H(V|_H, W)$ of intertwining operators. For example, let F be a local field and E/F be a quadratic extension. Let $\sigma \neq 1$ be the non-trivial element in $\mathrm{Gal}(E/F)$. Let $G = \mathrm{GL}_2(E)$ and $H = \mathrm{GL}_2(F)$. Let (π, V) be an irreducible smooth and admissible representation of G and let χ be a character of $F^\times$. Put

$$\mathrm{Hom}_H(\pi|_H \otimes \chi^{-1}, \mathbf{C}) := \{\lambda : V \to \mathbf{C} \mid \lambda(\pi(h)v) = \chi(\det h)\lambda(v) \text{ for all } h \in H,\, v \in V\}.$$

We say that π is χ-distinguished by H if $\mathrm{Hom}_H(\pi|_H \otimes \chi^{-1}, \mathbf{C}) \neq 0$. Let $\tau_{E/F}$ be the quadratic character of $F^\times$ associated with E/F. A classical result asserts that

Theorem 1.1. *The following are equivalent*

1. *π is either $\mathbf{1}$-distinguished or $\tau_{E/F}$-distinguished by H*
2. *$\pi^\sigma \simeq \pi^\vee$.*

Moreover, we have

$$\dim_{\mathbf{C}} \mathrm{Hom}_H(\pi|_H \otimes \chi^{-1}, \mathbf{C}) \leq 1, \quad \chi = \mathbf{1}, \tau_{E/F}.$$

There is a global counterpart of the above result. Let F be a number field and E/F be a quadratic extension. Let χ be a quadratic Hecke character of $\mathbf{A}_F^\times$ and let π be a unitary cuspidal automorphic representation of $\mathrm{PGL}_2(\mathbf{A}_E)$.

Theorem 1.2. *The following are equivalent:*

1. *The Asai L-function $L(s, \mathrm{As}(\pi) \otimes \chi)$ has a pole at $s = 1$.*
2. *π is χ-distinguished by $\mathrm{GL}_2(\mathbf{A}_\mathbf{Q})$, i.e. the period integral*
$$\int_{\mathbf{A}_F^\times \mathrm{GL}_2(\mathbf{A}_F) \backslash \mathrm{GL}_2(\mathbf{A}_F)} \phi(h)\chi(\det h)\mathrm{d}h \neq 0$$
for some $\phi \in \pi$.

When either of the above conditions is satisfied,

$$\dim_{\mathbf{C}} \mathrm{Hom}_{\mathrm{GL}_2(\mathbf{A}_F)}(\pi|_{\mathrm{GL}_2(\mathbf{A}_F)} \otimes \chi^{-1}, \mathbf{C}) = 1.$$

In this project, we aim to extend the above results to the following cases

Date: March 16, 2025.

(A) $G = D_E^\times$ and $H = D^\times$ or the unitary group $U(2)$. Here $D \neq \mathrm{M}_2(F)$ is a quaternion algebra over F.
(B) $G = \widetilde{\mathrm{SL}}_2(E)$ be the double cover of $\mathrm{SL}_2(E)$ and $H = \mathrm{SL}_2(F)$.

2. The plan

- In the first two weeks of the program, we will introduce a general strategy to study invariant forms via the involution method. After working out details of the papers [Hak91] and [Mat09], participants are expected to apply the Gelfand-Kazhdan criterion to obtain a proof of Theorem 1.1 as well as obtain generalizations in the case (A), including the archimedean case.
- In the third week, we investigate the case $(G, H) = (\mathrm{SL}_2(E), \mathrm{SL}_2(F))$. Study the details of [AP03] to understand the relation between the dimension of $\mathrm{Hom}_H(\pi, \mathbf{C})$ and the L-packets of $\mathrm{SL}_2(F)$ and try to imitate the method to work out the generalization of the case (B).
- In the fourth week, we turn to the global case. The goal is to understand Eisenstein series, the Rankin-Selberg convolution and the proof of Theorem 1.2.
- The Rankin-Selberg method cannot be applied to the case (A) for there is no Eisenstein series on $D^\times$ if $D \neq \mathrm{M}_2(F)$. In the fifth week, we introduce global theta lifts to study this case.
- In the last week, we consider the global case $(G, H) = (D_E^\times, D^\times)$. Let π be a unitary irreducible automorphic representation of $PG(\mathbf{A})$ with
$$\pi^\sigma = \pi.$$
Realizing an automorphic form $\phi \in \pi$ as a theta lift from $\mathrm{GL}_2(\mathbf{A})$, we can express the period integral
$$P_H(\varphi) = \int_{\mathbf{A}^\times H(\mathbf{Q})\backslash H(\mathbf{A})} \phi(h)\mathrm{d}h$$
in terms a product of certain local zeta integrals. Finally evaluate explicitly these local zeta integrals and obtain an identity between $P_H(\varphi)$ and the residue of the Asai L-function $L(s, \mathrm{As}(\pi))$ at $s = 1$.

Shih-Yu Chen (NTHU) and Yao Cheng (TKU) will collaborate with me in USRP. Y. Cheng and S.-Y. Chen will work with students on local and global aspects respectively.

3. Prerequisite and References

Prospective students for this proposal are expected to master the following tools:

- Galois theory, representation theory of finite groups and theory of semisimple rings.
- Basic number theory at the level of [Ser73, Part I].
- Fourier analysis on adeles of number fields (Tate's thesis)[Fro67, Chapter XV].
- Representation theory of GL(2) over local fields at least the level of [Bum97, Chapter 4, page 397-523]. It would be great if students can have a basic understanding about the material in [Gel75, §3-7].

Students with the above background need to study the notes from my course **Weil representation and automorphic forms** given this semester to prepare the global theory before the start of USRP.

References

[AP03] U. K. Anandavardhanan and Dipendra Prasad, *Distinguished representations for* SL(2), Math. Res. Lett. **10** (2003), no. 5-6, 867–878. MR 2025061

[Bum97] D. Bump, *Automorphic forms and representations*, Cambridge Studies in Advanced Mathematics, vol. 55, Cambridge University Press, Cambridge, 1997.

[Fro67] *Algebraic number theory*, Proceedings of an instructional conference organized by the London Mathematical Society (a NATO Advanced Study Institute) with the support of the Inter national Mathematical Union. Edited by J. W. S. Cassels and A. Fröhlich, Academic Press, London, 1967.

[Gel75] Stephen S. Gelbart, *Automorphic forms on adèle groups*, Annals of Mathematics Studies, No. 83, Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1975. MR 379375

[Hak91] Jeff Hakim, *Distinguished p-adic representations*, Duke Math. J. **62** (1991), no. 1, 1–22. MR 1104321

[Mat09] Nadir Matringe, *Distinguished principal series representations of* $\mathrm{GL}(n)$ *over a p-adic field*, Pacific J. Math. **239** (2009), no. 1, 53–63. MR 2449011

[Ser73] J.-P. Serre, *A course in arithmetic*, Springer-Verlag, New York, 1973, Translated from the French, Graduate Texts in Mathematics, No. 7.