Proposal for 2025 NCTS USRP

Title: Exploring the boundary of spaces of varieties

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Algebraic varieties are the main objects of investigation in algebraic geometry. A natural and important questions is the classification of algebraic varieties. Roughly speaking, we are interested in understanding how many varieties are there? This leads to the study of various geometric invariants, the properties of these invariants, and the properties of the varieties with extremal invariants. The purpose of this program is to introduce some modern aspects of research of algebraic geometry via various explicit examples. We will focus more on interesting and workable examples and hoping that students in this program will get better idea of the motivations of notions in algebraic geometry.

The topics that we are going to work on can be categorized into the following three parts.

1. DEGENERATION OF VARIETIES (AS BOUNDARY OF MODULI SPACES)

Usually, there are "many" varieties with the same invariants. It is quite often they forms a family varying continuously and there is a parameter space parametrizes the family. Once there is such family, varieties might degenerate to some more special varieties. It is interesting to look at some of the examples and considering possible technique to handle them. Below are some typical examples:

a. Computation of degeneration of spaces attached to K3
   1. V. Kulikov, Degenerations of K3 surfaces and Enriques' surfaces. Math. USSR Ilv. 11, 957- 989 (1977)
   2. Persson, U.: On Degenerations of Algebraic Surfaces. Mere. Am. Math. Soc. 189 (1977)
   3. R. Friedman and F. Scattone, Type III degenerations of K3 surfaces. Inventiones Math. (1986)

b. Degeneration of surface of general type
   1. M. Franciosi, R. Pardini, J. Rana, S. Rollenske, I-surfaces with one T-singularities, Bollettino dell'Unione Matematica Italiana (2022)

2. W. Liu and S. Rollenske, Geography of Gorenstein stable log surfaces, TAMS (2016)
3. M. Franciosi, R. Pardini, S. Rollenske, Log-canonical pairs and Gorenstein stable surfaces with K^2=1, Comp. Math. (2015)

2. COMPUTATIONS OF VARIOUS THRESHOLDS (AS BOUNDARY OF INVARIANTS OF PAIRS)

To study the geometry of varieties, it is also natural to study subvarieties inside a given variety in question. Subvarieties of codimension 1 (and combination of them) are called divisors. Subvarieties or divisors with same invariants form a family. General members in such family are usually smooth and nice to handle. The most singular ones are critical. How singular they can be is governed by the nature of the ambient varieties in question. Such kind of consideration is the foundation of minimal model program. Below are some typical examples:

a. log canonical thresholds
   1. Cheltsov, I.: Log canonical thresholds on hypersurfaces. Sb. Math. 192, 1241–1257 (2001)
   2. Cheltsov, I.: Log canonical thresholds of del Pezzo surfaces. Geom. Funct. Anal. 18, 1118–1144 (2008)
   3. Ahmadinezhad, H.; Cheltsov, I.; Schicho, J.: On a conjecture of Tian. Math. Z. 288 (2018), no. 1-2, 217–241.

b. canonical threshold
   1. Corti, Alessio: Singularities of linear systems and 3-fold birational geometry. Explicit birational geometry of 3-folds, 259–312. London Math. Soc. Lecture Note Ser., 281 Cambridge University Press, Cambridge, 2000.
   2. Pukhlikov, Aleksandr: Birationally rigid Fano-Mori fibre spaces. Forum Math. Sigma 12 (2024), Paper No. e84, 63 pp.
   3. Krylov, Igor; Okada, Takuzo; Paemurru, Erik; Park, Jihun: 2n^2-inequality for cA1 points and applications to birational rigidity. Compos. Math. 160 (2024), no. 7, 1551–1595.

c. (log)-canonical threshold of hypersurface singularities
   1. Kuwata, Takayasu: On log canonical thresholds of surfaces in C^3. Tokyo J. Math. 22 (1999), no. 1, 245–251.
   2. Prokhorov, Yuri: Gap conjecture for 3-dimensional canonical thresholds. J. Math. Sci. Univ. Tokyo 15 (2008), no. 4, 449–

459.

3. GEOGRAPHY OF VARIETIES (AS BOUNDARY OF INVARIANTS OF VARIETIES)

Geometric invariants are core of the investigation especially in classification theory. It is thus also important to know the distribution of invariants. Or if there are any relations between various invariants. The geography problems aim to consider the distribution of invariants and to investigate those objects with extremal invariants. Below are some examples that we may consider in our program:

a. Fano varieties with the large invariants
   1. K. Watanabe, Fano varieties with large pseudoindex and non-free rational curves, arXiv (2024)
   2. C. Wang, Fano varieties with conjecturally largest Fano index, IJM (2024) arXiv
   3. Z. Zhu, Volumes of Fano Manifolds, pdf
   4. I. Karzhemanov, Fano threefolds with canonical Gorenstein singularities and big degree. Math. Ann. 362, 1107–1142 (2015)
   5. Z. Zhuang, Fano varieties with large Seshadri constants, Adv. in Math. (2018)

b. Surfaces on the Noether line
   1. E. Horikawa, Algebraic Surfaces of General Type With Small c^2_1, I, Annals of Mathematics, Vol. 104, No. 2 (Sep., 1976), pp. 357-387
   2. Horikawa Surfaces, J. Evans

In the first two weeks of the program, we will focus on some basic notions and ideas of algebraic geometry. Then we will expect students to get their hands dirty by working on a small project in the middle.

In the latter half of the program, we will introduce some different examples based on the notions we introduced. Then we expect students will be able to handle more delicate examples at the end of the program.

ARRANGEMENTS

During the six-week USRP, except some TA sessions in the afternoons, students are expected to conduct constantly and continuously group study and independent study.

Week 1: Four 1~1.5 hour lectures

Lecture 1. Riemann-Roch Formula on Curves and Surfaces
Lecture 2. Intersection number
Lecture 3. Linear system of divisors
Lecture 4. Blow up and resolutions (of singularities and maps)

Week 2: Four 1~1.5 hour lectures

Lecture 1. Birational equivalence
Lecture 2. Singularities I: Quotient and ADE singularities
Lecture 3. Singularities II: Discrepancies (cones) and cyclic quotients
Lecture 4. Singularities III: Log pairs and log canonical thresholds

Week 3 and 4: Four 1~1.5 hour lectures

a. Instructors will give another 4 lectures in the morning
b. Term-Project will be assigned at the end of the third week
c. Students presentations
d. Additional meeting time supervising the student projects will be allocated.

Week 5 and 6: Four 1~1.5 hour lectures

a. Instructors will give another 4 lectures in the morning.
b. Students presentations
c. Additional meeting time supervising the student projects will be allocated.

INSTRUCTOR

Jungkai Alfred Chen, Professor, NTU/NCTS
Ching-Jui Lai, Associate Professor, NCKU
Jheng-Jie Chen, Assistant Professor, NCU

TEACHING ASSISTANT

TBD

PREREQUISITES (學生應備背景知識):

Students can start with only knowing projective varieties. One can pick up the basics from

a. Localization, Hilbert Nullstellensatz, valuation ring, DVR, integral closedness
b. Affine and projective varieties

PRE-PROGRAM READING (加入活動前指定閱讀規劃**)**

Students should know more about singularities in MMP (in 2d, the general definitions, and examples), positivity of divisors (esp. intersection number), and get the idea of the MMP, but not necessarily master all the mentioned subjects.

a. Chapter II, III.5, and III.6 of M. Reid's UAG
b. Chapter I to V of Fulton's book on Algebraic Curves

MORE ADVANCED READING (NOT NECESSARY)

c. Chapter 1, 3, 4 of K. Matsuki's Introduction to Mori's Program
d. Chapter 2, 4 of Kollar's and Mori's "Birational Geometry of Algebraic Varieties"

REFERENCES

(1) Toric Varieties, by David Cox, John Little, and Hal Schenck
(2) Positivity in Algebraic Geometry, by Robert Lazarsfeld
(3) The Rising Sea: Foundations of Algebraic Geometry, by Ravi Vakil
(4) Complex Algebraic Surfaces by A. Beauville.
(5) Compact Complex Surfaces by W.P. Barth, K. Hulek, C.A.M. Peters, A. Ven
(6) Introduction to Mori Program by K. Matsuki
(7) Birational Geometry of Algebraic Varieties by J. Kollar, S. Mori