2024 NYCU-Math TA training

(1) **The definition of limits** Let $L \in \mathbb{R}$. We write

$$\lim_{x \to a} f(x) = L$$

if, for any $\epsilon > 0$, there is $\delta > 0$ such that

$$|f(x) - L| < \epsilon, \quad \forall 0 < |x - a| < \delta.$$

(2) **The intermediate value theorem** If $f : [a, b] \to \mathbb{R}$ is continuous, then, for any L between $f(a)$ and $f(b)$, there is $c \in (a, b)$ such that $f(c) = L$.

(3) **The extremum value theorem** If $f : [a, b] \to \mathbb{R}$ is continuous, then f attains its maximum and minimum values. That is, there are $\alpha, \beta \in [a, b]$ such that

$$f(\alpha) \le f(x) \le f(\beta), \quad \forall x \in [a, b].$$

(4) **The definition of differentiation** Let $a \in \mathbb{R}$. A function f is differentiable at a if

$$f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \quad \text{exists},$$

or equivalently if there is (unique) $M \in \mathbb{R}$ such that

$$\lim_{x \to a} \frac{|f(x) - (f(a) + M(x - a))|}{|x - a|} = 0.$$

In particular, $M = f'(a)$.

(5) **The mean value theorem for derivatives** Let f be a function defined on $[a, b]$. If f is continuous on $[a, b]$ and differentiable on (a, b), then there is $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(6) **The chain rule** If f is differentiable at x and g is differentiable at $f(x)$, then $g \circ f$ is differentiable at x and $(g \circ f)'(x) = g'(f(x))f'(x)$.

(7) **The inverse function theorem** Suppose $f : (a, b) \to \mathbb{R}$ is one-to-one and differentiable on (a, b). If $f'(x) \neq 0$, then f^{-1} is differentiable at $f(x)$ and

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

(8) <u>**The l'Hospital's rule**</u> Let f, g be differentiable functions defined on (a, b). Assume that

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) \in \{0, \pm\infty\}, \quad \lim_{x\to a^+} \frac{f'(x)}{g'(x)} \in \mathbb{R} \cup \{\pm\infty\}.$$

Then,

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}.$$

(9) <u>**The fundamental theorem of calculus**</u> Let $f : [a, b] \to \mathbb{R}$ be continuous.

(1) If $F(x) = \int_a^x f(t)dt$, then F is an antiderivative of f on (a, b) and continuous on $[a.b]$.

(2) If $G : [a, b] \to \mathbb{R}$ is an antiderivative of f on (a, b) and continuous on $[a, b]$, then $\int_a^b f(t)dt = G(b) - G(a)$.

(10) <u>**The mean value theorem for integrals**</u> If $f : [a, b] \to \mathbb{R}$ is continuous, then there is $c \in (a, b)$ such that

$$\int_a^b f(x)dx = f(c)(b - a).$$

(11) <u>**Lagrange multipliers**</u> Let $f, g_1, ..., g_k : \mathbb{R}^n \to \mathbb{R}$ be continuously differentiable functions with $k \le n$ and $E = \{x \in \mathbb{R}^n | g_i(x) = 0, \forall 1 \le i \le k\}$. Suppose $\nabla g_1(x), ..., \nabla g_k(x)$ are linearly independent for all $x \in E$. If f, restricted to E, attains its maximum or minimum at P, then there are constants $c_1, ..., c_k$ such that

$$\nabla f(P) = c_1 \nabla g_1(P) + \cdots + c_k \nabla g_k(P).$$

(12) <u>**Taylor series and analyticity**</u> Let f be a function which is infinitely differentiable at a. The Taylor series of f centered at a refers to the following series

(1) The Taylor series of f centered at a refers to the following series

$$T(x) := \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x - a)^n.$$

(2) f is analytic at a if there is $\epsilon > 0$ such that

$$f(x) = T(x), \quad \forall x \in (a - \epsilon, a + \epsilon).$$

(13) <u>**Fubini's Theorem**</u> Let D is a rectangle region be contained in $\mathbb{R}^2$

$$D = \{(x, y) \mid a \le x \le b,\ c \le y \le d\}$$

and let $f : D \subseteq \mathbb{R}^2 \to \mathbb{R}$ is integrable. If $f_y(x) = f(x, y)$ is integrable on $[a, b]$ for each $y \in [c, d]$, and if $g(y) = \int_a^b f(x, y)dx$ is integrable on $[c, d]$. Then the Riemann integral of f over D equals to the iterated integral

$$\int\int_D f(x,y)dA = \int_c^d \left(\int_a^b f(x,y)dx \right) dy$$

(14) <u>**Change of Variables for multiple integrals**</u> Given U is a open set in $\mathbb{R}^n$, let $g : U \to \mathbb{R}^n$ is one to one and continuously differentiable on U. If the Jacobian of g, $Jg(\mathbf{x}) \neq 0$ on U, if A is a Jordan measurable and $\overline{A} \subseteq U$, if f is bounded and integrable on $g(A)$, then $f \circ g$ is integrable on A and

$$\int_{g(A)} f(\mathbf{y})d\mathbf{y} = \int_A f(g(\mathbf{x}))\ |Jg(\mathbf{x})|\ d\mathbf{x}$$